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Monte Carlo Renormalization Group Calculations for SU(2) Lattice Gauge Theory*

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ABSTRACT

A Monte Carlo renormalization group calculation of the β function in SU(2) lattice gauge theory is presented, using the blocking transformation proposed by Swendsen. I outline a strategy for improving the action using Monte Carlo methods and present some preliminary results.

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Renormalization, for the lattice regulator as for any other regulator, means showing that a theory predicts a unique long wavelength physics independent of the details of the cutoff and finding how the bare parameters of the theory must be varied with the cutoff to keep the long wavelength physics constant. In the Monte Carlo renormalization group approach, developed by Ma, Swendsen and Wilson,[1,2] this is achieved by calculating vacuum expectation values on lattices of spacing a and 2a, and adjusting the bare parameters on the lattice of spacing 2a to get the same physics as on the lattice of spacing a. Monte Carlo computer simulations must be done on lattices of finite volume; this may strongly affect vacuum expectation values. The same physical volume is therefore used on both lattices to produce identical infrared effects on the quantities being matched. For example, in the calculations reported on here, quantities measured on a 4^4 lattice of spacing 2a were matched to those measured on an 8^4 lattice of spacing a.

In principle, quantities directly connected with physics, such as glueball correlation functions or the heavy quark potential, can be used for matching, but it is difficult statistically to study the long wavelength properties of these quantities. Therefore, a renormalization group transformation is performed on the 8^4 lattice to get a 4^4 lattice on which each new link is in some sense the average of neighboring links on the old lattice. The matching is done between quantities constructed of these "block links."

The kernel defining the transformation may be chosen arbitrarily. A variety of kernels has been suggested for lattice gauge theories.[1,3,4,5] By adding large numbers of terms to the action, equally good matching can presumably be obtained using a wide variety of

kernels. These various actions, which are equally good in predicting physical quantities correctly, will in general differ widely in computational simplicity. For example, decimation kernels in spin systems are known to lead to effective actions which are very nonlocal and hence difficult to use. For subsequent convenience, the kernel should be chosen to obtain the simplest possible effective action consistent with predicting the physics correctly. It is not to be expected that any choice of kernel, no matter how clever or elaborate, will lead to exactly the Wilson action at finite lattice spacing; this would imply that the Wilson action at finite lattice spacing predicts physical quantities with no errors due to the finite lattice spacing. However, it is desirable to find an effective action which, while getting the physics right, is as local as possible.

In QCD and similar theories, only one combination of operators in the action should be relevant at large distances. Therefore, after many iterations of the blocking procedure, the same physics should be obtained independent of the details of the bare parameters, depending only on the coefficient of the most relevant operator. If enough renormalization group transformations can be performed on the lattice sizes available, the beta function can be calculated by matching quantities at very large distance scales without improving the action (Swendsen[6]). In practice, on a finite lattice, some cutoff dependent effects will probably remain, even after 2,3, or 4 levels of blocking. These must be removed by improving the action, the kernel or both.

The calculations are done in the following way. Monte Carlo simulations are done on an 8^4 lattice with an action

$$S = \beta_1 \Sigma_{\text{lattice}} U_1 + \beta_2 \Sigma_{\text{lattice}} U_2 + \cdots ,$$

where the U_i are traces of simple loops (possibly in various representations of the gauge group) and the coefficients β_i are chosen arbitrarily. (I have used Creutz's[7] heat bath procedure for the simulations.) Renormalization group transformations are applied to the 8^4 lattice to obtain lattices of sizes 4^4, 2^4 and 1^4. Expectation values $\langle U_i^{(n)} \rangle$ are measured on these blocked lattices, where $U_i^{(n)}$ might be the trace of some simple loop at the nth level of blocking.

A trial run is made on a 4^4 lattice with new couplings guessed one way or another. The discrepancies between quantities at the (n)th level of blocking on the 8^4 lattice and the (n-1)st level of blocking on the 4^4 lattice are determined:

$$\Delta\langle U_i \rangle = \langle U_i^{(n)} \rangle_{8^4} - \langle U_i^{(n-1)} \rangle_{4^4} \neq 0.$$

The changes in the couplings on the 4^4 lattice necessary to correct the discrepancies may be estimated by calculating the matrix of derivatives,

$$\frac{d}{d\beta_j} \langle U_i^{(n-1)} \rangle = \langle U_i^{(n-1)} \sum U_j^{(0)} \rangle - \langle U_i^{(n-1)} \rangle \langle \sum U_j^{(0)} \rangle ,$$

and solving the first order equation

$$\Delta\langle U_i^{(n-1)} \rangle = \frac{d}{d\beta_j} \langle U_i^{(n-1)} \rangle \Delta\beta_j$$

for $\Delta\beta_j$.

Another simulation on the 4^4 lattice with the new β_j may be performed and the process iterated if necessary. If the statistics on the derivative matrix are good enough, no multiparameter search is necessary, even with multiparameter actions.[8]

As a sample calculation, I show results from a one parameter matching of the simple plaquette (Wilson) action, using Swendsen's kernel" (the simplest sensible kernel) to define the block variables. Results of a matching are shown in Table 1. The data are from four runs of 1536 iterations each on an 8^4 lattice at β=2.5 and four runs of 8092 iterations each on a 4^4 lattice at β=2.322. The errors quoted are the standard derivations of the four runs. The matching in the upper two levels of blocking is impressive, usually within one standard deviation, even though no effort has been made yet to optimize the kernel or the action. The matching at the lowest level is poor, as expected, since "irrelevant" operators may have large effects at short distance scales.

From a sequence of such matchings at different couplings β, a "β function" may be calculated. A graph of $\Delta\beta$ vs. β in the crossover region is shown in Fig. 1. At low β, the loops on the 1^4 lattices (top level of blocking) are dominated by finite volume effects and are virtually independent of β. Hence, a β function can be calculated on the basis of the middle level of blocking only. The sign and magnitude of this β function are respectable; it does not match onto the value expected from weak coupling perturbation theory at large β, however. There are two possible explanations. One is that the Monte Carlo calculation has not been carried to large enough β that we should expect to recover the perturbative result. A more likely explanation in view of the discrepancy in the $\Delta\beta$'s obtained from the top and middle levels

TABLE I

Loop expectation values ($1/2 \langle TrU_i \rangle$) calculated on an 8^4 lattice at $\beta=2.5$ (Wilson action) blocked n times and a 4^4 lattice at $\beta=2.322$ blocked n-1 times.

	Plaquette	(Plaquette)^2	Flat 6 Link	3 Dim 6 Link	Flat 2 × 2	4 Dim 8 Link
1^4 Block Lattices						
8^4 lattice at 3rd level of blocking	.747(11)	.645(11)	.686(11)	.615(16)	.630(31)	.525(22)
4^4 at 2nd level	.757(2)	.657(2)	.684(2)	.609(3)	.614(6)	.515(5)
2^4 Block Lattices						
8^4 at 2nd level	.589(4)	.447(3)	.374(9)	.344(5)	.378(17)	.212(8)
4^4 at 1st level	.589(2)	.449(2)	.390(2)	.346(2)	.398(3)	.214(2)
4^4 Block Lattices						
8^4 at 1st level	.654(1)	.502(1)	.426(1)	.434(1)	.200(1)	.299(1)
4^4 unblocked	.615(1)	.465(1)	.406(1)	.395(1)	.209(1)	.261(1)

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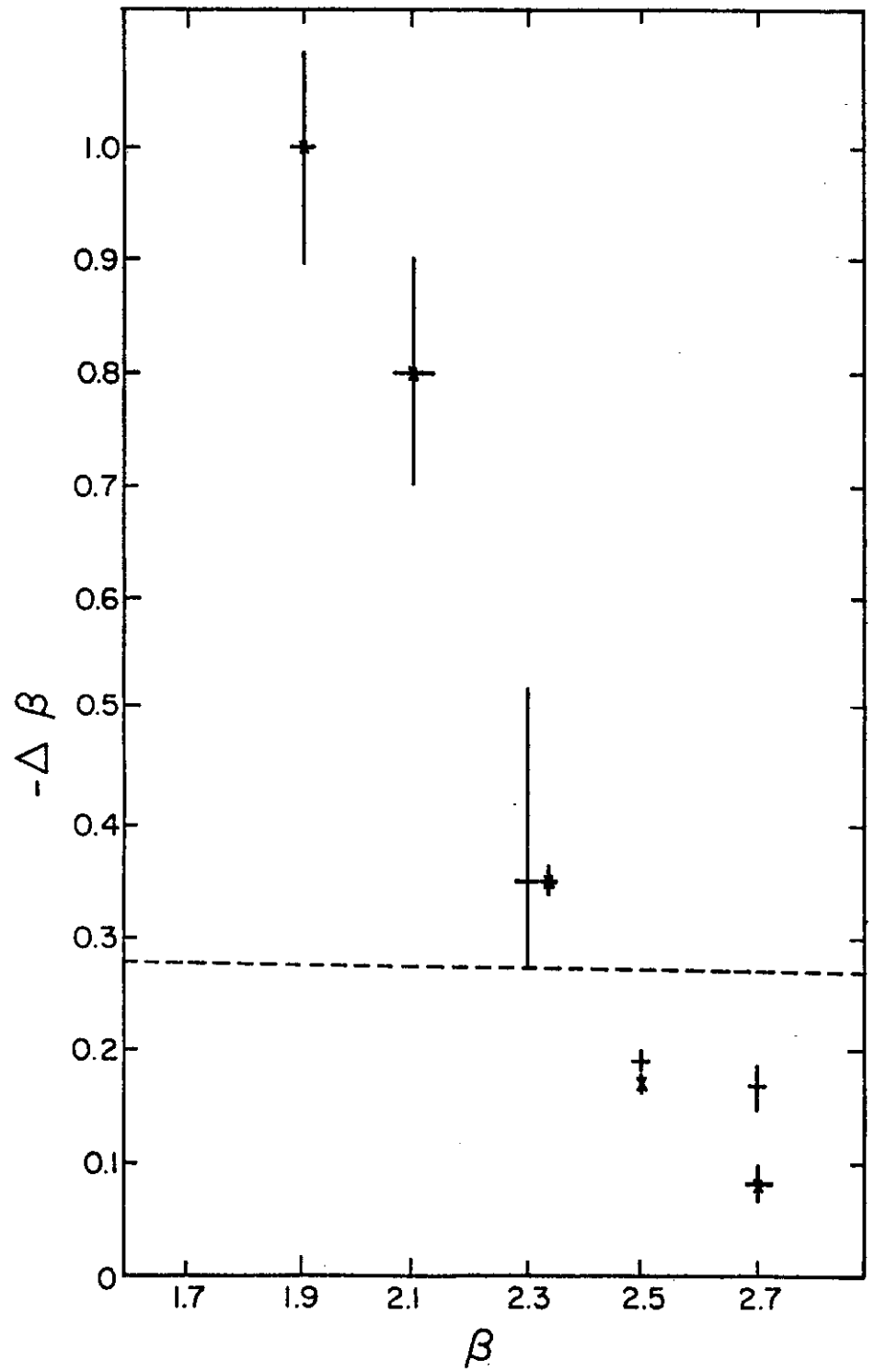

Fig. 1: The SU(2) β function in the crossover region for the Wilson action using the Swendsen kernel. $\Delta\beta$ for a scale change of 2 is plotted vs. β. The lines are results of matches to middle blocking level data. x's are from matches at the top level. The dashed line is the two loop perturbative prediction for $\Delta\beta$.

of blocking at the largest value of $\beta(\beta=2.7)$ is that the 8^4 lattice is too small to allow enough blocking to remove completely the effects of the irrelevant operators. In fact, the shape of the graph resembles that obtained in analogous calculations for the O(3) spin model. For this model, Shenker and Tobochnik[9] found that they were able to match onto the weak coupling results by performing the simulations with an improved three-term action, optimized to remove the effects of the leading irrelevant operators in weak coupling. Recently, Hasenfratz, Hasenfratz, Heller and Karsch[10] have been able to achieve this matching in a simpler way. Using weak coupling calculations they find an improved kernel which leads to a more local effective action and allows good matching using only the most local term in the effective action. Analogous perturbative calculations must be performed for the gauge theory and the Monte Carlo calculations must be carried to large enough β that the two methods can be matched.

In addition to these essential weak coupling calculations, one would like to have purely nonperturbative means for determining optimum renormalization procedures. In practical Monte Carlo calculations in the crossover region, "asymptotic" scaling of dimensionful quantities with β according to the weak coupling formula is often used as a test for the absence of finite lattice spacing effects. However, nonperturbative contributions to the β function may cause deviations from asymptotic scaling in this region even if finite a effects are small (that is, even if the renormalization transformation is nearly perfect.) On the other hand, when the size of the links becomes much larger than the size of the hadronic physical states, the description of the physics in terms of colored quark and gluon fields, while

mathematically possible, most likely becomes very awkward. Crazily nonlocal actions may be required to represent accurately the hadron physics. In ref. 9, a breakdown in matching at high temperatures was observed in the O(3) spin model using an action improved by weak coupling methods. A change of variables to fields connected with the physical states at large distance scales may be appropriate.[11] It is desirable to determine nonperturbatively the deviation of the optimized renormalization procedure from the weak coupling results and the possible breakdown of the simple procedure.

In the remainder of the talk, I would like to investigate the extent to which an improved action may be calculated at intermediate coupling by Monte Carlo methods. A different strategy is required for this than for the β function calculation. For the β function, it is best to match quantities calculated at the largest scale possible, to minimize the effects of the irrelevant operators. But statistical errors are worst at large distance scales, making it difficult to find and remove the suppressed effects of the irrelevant operators. On the other hand, at the lowest level of blocking, statistics are very good and the effects of the irrelevant operators are not attenuated. A very large number of very irrelevant operators which have negligible effect on any larger distance scale may be important at very short distances. Attempting to match the effects of these operators with a small number of parameters in the action may get wrong the operators relevant at larger distances. A strategy for matching with a finite number of parameters is thus: try the matching at a scale such that a small number (≥ 1) of irrelevant operators are visible over the statistics. Tree diagram calculations suggest that a matching with a finite number of

parameters may be possible at the second level of blocking but not at the lowest level. A crucial consistency condition for this type of approach is that the results of the matching get the physics right at all larger distance scales.

As an example, I will show the results of a two parameter matching of the second level block loops on the 8^4 lattices from table I. The two terms in the action are the plaquettes and the flat six link loops: $S = \beta_1 \Sigma U_p + \beta_2 \Sigma U_6$. The plaquette and the flat six link loop at the second blocking level are used to fix the parameters in the action. Two questions must now be answered. First, can a statistically meaningful result be obtained at all from the data? It may appear from table I that no discrepancy in the middle level is more than about one and a half standard deviations, and that therefore the discrepancies are meaningless. This is, in fact, not true. The errors in the various loops are very highly correlated, and the fact that, for example, the flat six link loop in the 4^4 lattices is a little high compared to the plaquette is very stable, run after run. A careful analysis of the errors in the β's obtained is required and was done in the following way. Appropriate β's giving the desired matching were obtained by inverting the derivative matrix from a guessed starting place, as described above. From this new starting point, a sequence of three runs on the 4^4 lattice was made. Each of these was compared with each of the four separate runs on the 8^4 lattices, and the small corrections in β necessary to achieve matching were calculated for each of the 12 combinations. The spread in β's obtained is shown in Fig. 2, and yields $\beta_1 = 2.95 \pm .12$ and $\beta_2 = -.162 \pm .035$. Note from the graph that one combination of the β's has a dominant effect on the physics and is known

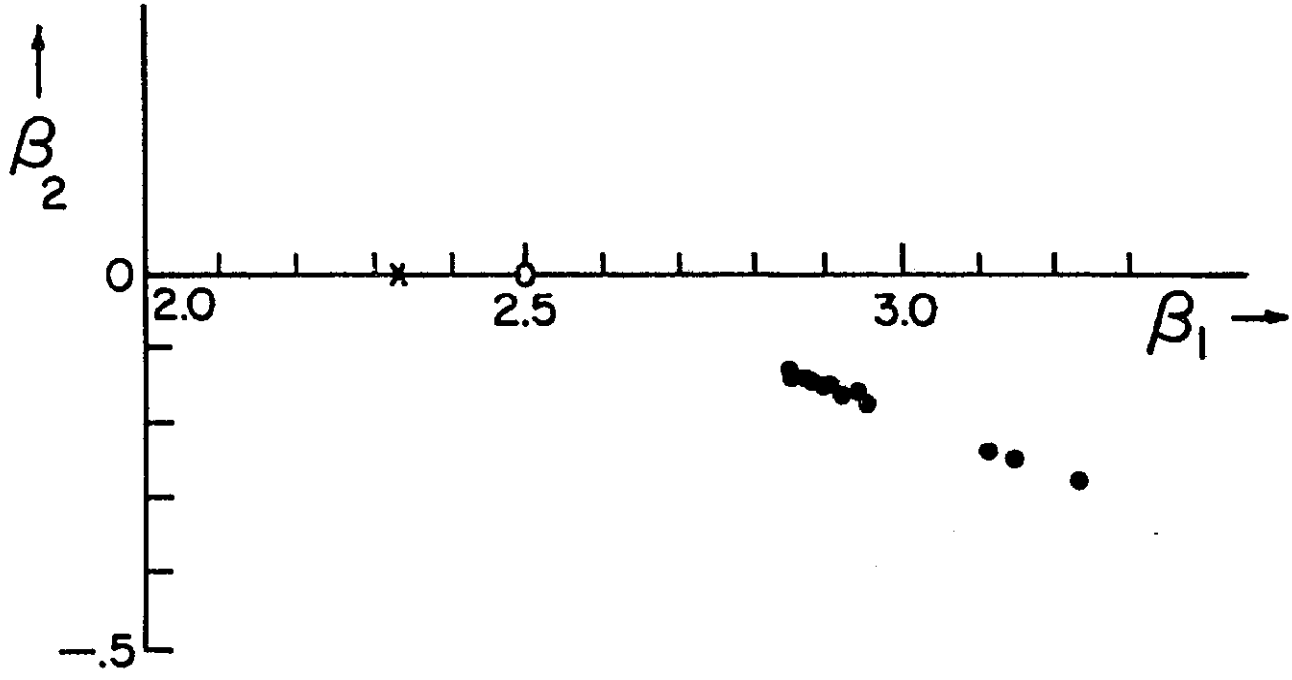

Fig. 2: Couplings obtained by matching a two parameter action on a 4^4 lattice to the data at the middle blocking level on an 8^4 lattice at $\beta=2.5$. The circle is the original action on the 8^4 lattice. The x is the renormalized coupling $\beta=2.322$ from a one parameter fit on a 4^4 lattice. The twelve dots are the results of a two parameter fit on a 4^4 lattice from various combinations of runs.

almost perfectly, while the other combination is poorly determined. Even so, the coefficient β_2 of the six link loops can be obtained to an accuracy of 20%.

The second question to answer is, if the procedure yields good statistics, does it also yield good physics; does this fit help or hurt the fit of all other loops, especially these at highest blocking level? Table II contains some of the results from the one parameter fit of table I plus results for the two parameter fit just described. The quantities that were badly off in the one parameter fit now are in good agreement, about 1%, while the quantities which agreed well before are still in agreement. For comparison, results from a two parameter fit to data at the lowest blocking level are also included. Fits at this level are trivially easy to do; the statistics are great on both the expectation value and the derivative matrices. However, even though the fit at the lowest level is very easy and works well at that level, the matching at the higher blocking level is terrible, for reasons explained above.

In summary of the results obtained so far: 1) Long wavelength Monte Carlo renormalization group matching is pretty good (though not perfect) even using a nonoptimized blocking transformation and a one parameter plaquette action. 2) Multiparameter MCRG matching of short wavelength data is very easy, but may well get the long wavelength physics wrong. However, it is possible with existing computing power to get useful information about nonleading operators from intermediate wavelength data.

TABLE II

The block data of Table I is fit on 4^4 lattices with two parameter fits in two different ways. A two parameter fit to data at the middle level of blocking improves the matching. A two parameter fit to data at the lowest level of blocking, while working well for the data at that level (not shown), ruins the fit at larger distance scales.

	Middle level of Blocking					Top level of blocking
	Plaquette	Flat 6 Link	3 Dim 6 Link	Flat 2 x 2	4 Dim 8 Link	Plaquette
8^4 Lattice $\beta = 2.5$	.589(4)	.374(9)	.344(5)	.377(17)	.212(8)	.747(11)
4^4 Lattice $\beta = 2.322$	.589(2)	.390(2)	.346(2)	.398(3)	.214(2)	.757(2)
4^4 Lattice $\beta_1 = 2.95$, $\beta_2 = -.167$ Fit to middle level of blocking.	.589(1)	.374(2)	.340(5)	.373(5)	.209(6)	.738(4)
4^4 Lattice $\beta_1 = 3.349$, $\beta_2 = -.2839$ Fit to lowest level of blocking.	.578(3)	.352(5)	.322(5)	.356(4)	.188(5)	.707(6)

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There are two types of ultimate goals for these calculations, fundamental and practical. Showing the renormalizability of lattice regulated QCD by direct calculation and recovering the weak coupling β function is of tremendous fundamental importance, even though we believe we know how the results of those calculations will turn out when they succeed. In addition, calculation of the β function in the intermediate coupling region may be very useful in interpreting scaling tests of phenomenological calculations. Beyond this, there is the open question of whether improved actions may ultimately be of any direct practical use in reducing finite lattice spacing errors more efficiently than simple lattice spacing reduction in phenomenological calculations. For the pure gauge theory at weak enough coupling this is almost certainly the case. For small a, finite lattice spacing errors go like $c_2a^2\Lambda^2+c_4a^4\Lambda^4+\ldots$, where a is the lattice spacing, Λ is the QCD scale and C_i depend on the quantity being considered. Reducing the lattice spacing by a factor of two costs a factor of 16 in CPU time (times an unknown factor for "critical slowing down") and reduces the finite a errors by about a factor of four. If the $a^2\Lambda^2$ term in the errors can be removed by using a three loop action, costing perhaps a factor of eight over the plaquette action in CPU time (times another unknown factor), this reduces finite a errors by a factor of $a^2\Lambda^2$. At small enough a, the second procedure is almost certainly the most efficient. At the intermediate lattice spacings at which present day calculations are done, however, the answer is not obvious. An improved quark action, on the other hand, may be quite practical, even at intermediate coupling. The reason is that quark calculations employ relaxation algorithms. It may therefore be possible to use the simplest fermion action to get a

good approximation to the propagator, and to use the expensive extended action for finishing touches. Furthermore, it makes sense to consider improving the quark action before improving the glue action. The lowest dimension "missing" operators for the quarks are of dimension five rather than dimension six. At intermediate coupling, locality may be a more sensible classification scheme than dimension of operators; again, the operators which can be added to the quark action are more local than those which can be added to the glue action.

I have enjoyed conversations with Anna Hasenfratz, Michael Peskin and Steven Shenker.

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